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FLOW OVER AN AXISYMMETRIC BODY IN A CYLINDRICAL TUNNEL.(U)

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J. Fernandez

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The flow past an axisymmetric body in a cylindrical tunnel is of practical importance in the study of wind or water tunnel blockage effects and in the analysis of high speed tube transportation vehicles. The present study employs the Green's Function method to analyze the flow past arbitrary axisymmetric bodies in cylindrical tunnels. Towards this end, the appropriate fundamental solution to the Laplace's equation is developed. This solution automatically satisfies the boundary conditions at the tunnel wall and at the

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20. upstream and downstream infinities. The boundary condition at the body is formulated into an integral equation of the second kind. A numerical method is employed for the solution of the integral equation. It is shown further that the limiting case of the flow in a tunnel of infinite radius agrees very well with the previously known flow in an unbounded medium. The present method has been employed for the analysis of various body shapes of interest. The results show excellent agreement in cases when exact analytical solutions are available. The present method has also been found to be computationally efficient.

Subject: Flow Over an Axisymmetric Body in a Cylindrical Tunnel

References: See page 38.

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Table of Contents

Nomenclature

a	tunnel radius
$E(m)$	Complete Elliptic integral of the second kind
$F_o(k)$	finite Hankel transform of order zero
$F_1(s,s')$	axial velocity at s due to a unit source at s'
$F_2(s,s')$	radial velocity at s due to a unit source at s'
$g(P,P')$	regular part of the Neumann function
H_n	the integral (A-30)
$I_n(x)$	modified Bessel Function of the first kind and order n
j_m	m-th positive root of $J_1(w) = 0$
$J_n(x)$	Bessel Function of the first kind and order n
k_m	m-th positive root of $J_o'(ka) = 0$
$K(m)$	Complete Elliptic integral of the first kind
$K_n(x)$	modified Bessel Function of the second kind and order n
ℓ	total arc length of the body along the meridian
m	modulus of the elliptic integrals
$N(P,P')$	singular part of the Neumann function
O	order symbol
P	field point or observation point
P'	source point
$P_n(r,r')$	function defined by (A-17)
$Q_n(r,r')$	function defined by (A-28)
r	radius
s	arc length parameter

Nomenclature (cont.)

$V(P)$	single layer potential at point P
$W(P)$	double layer potential at point P
x	distance $\|z-z'\|$
$Y_n(x)$	Bessel Function of the second kind and order n
z	axial distance
α	angle between z-axis and the normal to the body pointing into the fluid
$\delta_p(P')$	Dirac delta function
Δ_o	operator defined by eqn. (21)
∇^2	axisymmetric Laplacian operator
Φ	velocity potential
$\Gamma(P,P')$	Neumann function
μ_{2n}	constants given by (A-18)
σ	source strength
Ξ_1, Ξ_2	sum of series (31) and (32)

Superscript

'	prime refers to the source point, except if used over a function it refers to differentiation with respect to the argument

Subscript

r	differentiation with respect to r
z	differentiation with respect to z

SECTION I

INTRODUCTION

1.1 Aim

The flow past an axisymmetric body in a cylindrical tunnel is of practical importance in the study of wind or water tunnel blockage effects and in the analysis of high speed tube transportation vehicles. This is a classical problem for which exact solutions are available only for simple body shapes such as spheroids [1, 2]. Some of the current approximate methods for the solution of this problem are applicable only to bodies of simple shape and can not deal with bodies with flat noses and bumps [6-9]. A few other methods satisfy the boundary condition at the tunnel wall only approximately and are not suitable for large blockages [11-14]. The present effort was therefore undertaken in order to develop an efficient analytical method applicable to the flow past arbitrary axisymmetric bodies in cylindrical tunnels.

1.2 Earlier Studies

The earliest analysis of the flow over a body in a tube was performed by Lamb [1] for a Rankine ovoid. Lamb obtained the potential and stream function for a source placed on the axis of the tube. By combining a source and a sink of equal strength in the presence of a uniform stream, he obtained the shape of the ovoid. Lock [2] obtained an analogous solution for the case of a Rankine ovoid in a open jet tunnel of circular cross section. The solutions of Lamb and Lock contained certain infinite series, which converged slowly near the singularity and failed to converge at the singularity itself. At the request of Lock, Watson [3] derived alternate forms of the solution

valid near the singularity with excellent convergence properties. Subsequently, Lock and Johansen [4] obtained wind tunnel interference corrections for a Rankine ovoid using Watson's result.

The velocity potentials and stream functions for various singularities inside a tube such as ring and point sources, doublets and vortices were derived by Levine [5]. However, Levine's results also contained infinite series similar to those of Lamb and Lock and, therefore, suffered from the same convergence problems. Alternate formulations for these potentials involving infinite integrals of the modified Bessel functions were employed by many authors. The integral form for the potential due to an axial source was used by Satija [6], who obtained first-order wall corrections for the flow over a spheroid; and Goodman [7] who applied slender body theory to analyze the flow about a body of revolution traveling in a tube. Landweber and Gopalakrishnan [8] used axial doublets and solved a pair of integral equations for determining the flow past a body in a tube. In a subsequent work, Landweber [9] observed that the basic potential for a source as used by Satija and Goodman was in the form of a divergent integral and corrected it. He also derived the potentials and stream functions for axial and ring sources, doublets and vortices in terms of integrals of the modified Bessel functions. He used axial sources to express the stream surface condition on the body as an integral equation of the first kind and examined three methods of solution.

It should be noted that the above methods use axial singularities in their analyses. It is known that methods using axial singularities can not deal with bodies of complex geometry such as those with flat noses and bumps [10]. The methods employing surface singularities are better suited for these

classes of bodies. An obvious surface singularity technique is to approximate the infinite tube by a finite tube of sufficient length and distribute surface singularities on the body as well as the tube. The Klein and Mathew [11] method employs this technique using ring vortices, while the Hess and Smith [12] method uses ring sources. The boundary condition at the wall is satisfied approximately by requiring the normal velocity to be zero at the control point for each surface element on the wall [12]. The normal velocity elsewhere on the wall need not vanish and therefore the fluid appears to "leak" through the wall. The effect of this leakage becomes significant as the blockage ratio (ratio of the maximum body radius to the tunnel radius) increases and the accuracy of these methods becomes poor. One technique to reduce this leakage is to require that the integral of the normal velocity at each element (or, equivalently, the mass flow across each element) be zero [11, 13].

In all these methods, the approximation of the infinite tube by a finite tube requires some care. In most cases, the tube is continued a few body lengths fore and aft of the body so that the velocities at the ends of the tube become sufficiently uniform. A recent analysis by Varsomov and Haimov [14] employs source discs at the ends of a finite tube to satisfy the uniform velocity condition at infinity. In any case, these finite-tube methods are approximate and require extra computational effort as well.

A surface singularity method satisfying the wall condition exactly is given by Miloh [15], who used ring vortices and obtained the exact solution for the case of a prolate spheroid. Miloh's method employed an integral equation of the first kind. Mathew and Majhi [16] used ring vortices and obtained an integral equation of the second kind for the boundary condition on the body. Their method satisfied the boundary condition on the wall exactly.

1.3 Scope of the Present Study

In the present study, we employ the Green's Function method to analyze the flow past arbitrary axisymmetric bodies in cylindrical tunnels. Towards this end, we develop the appropriate fundamental solution to the Laplace's equation. This solution automatically satisfies the boundary conditions at the tunnel wall and at the upstream and downstream infinities. The boundary condition at the body is formulated into an integral equation of the second kind. We present a numerical method for the solution of the integral equation. This method employs parabolic surface elements with linearly varying source strengths. We further show that the limiting case of the flow in a tunnel of infinite radius agrees very well with the previously known flow in an unbounded medium. Finally, we employ the present method for the analysis of various body shapes of interest. The results show excellent agreement in cases where exact analytical solutions are available.

In summary, the present method satisfies the boundary condition at the wall exactly. It also employs a tunnel of infinite length where the boundary conditions at the upstream and downstream infinities are satisfied exactly. Since it uses a surface singularity technique, it is capable of dealing with bodies of arbitrary shape, unlike the methods employing axial singularities [6, 8, 9]. Also, the present method uses an integral equation of the second kind and therefore is better than the methods using integral equations of the first kind in several respects [9, 15]. Some of these considerations are the computational stability, ease and accuracy of solution and the existence of the solution for arbitrary normal velocities on the body. The last configuration is of significance in the analysis of the boundary layer on the body. In all the above aspects the present method is comparable to that of Mathew and Majhi [16].

However, their formulation led to the numerical evaluation of double integrals involving considerable computational effort. The present method is capable of dealing with body shapes of practical interest and has been found to be computationally efficient.

SECTION II

PROBLEM FORMULATION

In this section we present the mathematical formulation for the flow past an arbitrary axisymmetric body in a tunnel. We review the rudiments of potential theory in order to obtain the solution of the axisymmetric Laplace equation by means of the fundamental solution appropriate to the region interior to the tunnel. This fundamental solution, also known as the Neumann function, satisfies the governing differential equation and the boundary conditions at the wall as well as the upstream and downstream infinities. The remaining boundary condition on the body surface is formulated into an integral equation of the second kind. It is observed that the kernal of the integral equation has poor convergence near the singularity while its convergence is excellent elsewhere. Therefore, special formulas, valid near the singularity, are developed for the kernal of the integral equation. These formulas are rapidly convergent and are well suited for computation.

2.1 Potential Theory

With the advent of high speed digital computers classical potential theory has found wide use in the solution of practical flow problems [10]. This theory expresses the solution of the Laplace equation for a given region in terms of certain singularity distributions on the boundary of the region. The potential theoretic solution has two distinct advantages: (i) it is applicable to very general shapes of the region and (ii) the resultant computations need be performed only on the boundaries of the region. This is in sharp contrast with the finite-difference techniques which, even though applicable to regions of arbitrary shapes, require computations throughout the region. We shall presently introduce the concept of the fundamental solution to the Laplace

equation and show how the fundamental solution and the appropriate surface source distribution can be used to solve the Neumann problem.

Let S_1 be a closed, axisymmetric surface placed in an infinitely long cylindrical tunnel, S_2 [Figure 1]. Let D_1 be the region bounded by S_1. Consider the region D between S_1, the body, and S_2, the tunnel. Let us seek the solution of the potential equation

$$\nabla^2\Phi(P) = 0 , P \in D , \tag{1}$$

where ∇^2 is the axisymmetric Laplacian operator.

A fundamental solution of the above differential equation is defined as the potential at any point P due to a unit source at an arbitrary point $P' \neq P$, such that the boundary conditions at the tunnel wall as well as at upstream and downstream infinities are satisfied. Throughout this work we will follow the convention whereby the source point is denoted by a primed entity and the observation point (or the field point) is denoted by an unprimed entity. The only exception to this convention would be when the primed entity is a function, in which case it denotes the derivative of the function with respect to its argument.

Suppose a fundamental solution Γ is given, defined on the region $D \cup D_1 = M$ and satisfying the equation

$$\nabla^2 \Gamma (P, P') = -\delta_P(P') \; ; \; P, P' \in M , \tag{2}$$

where $\delta_P(P')$ is the Dirac delta function.

Cross-multiplying Γ and Φ and applying Green's theorem, viz:

$$\int_D (u\nabla^2 v - v\nabla^2 u)d\tau = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS \quad , \tag{3}$$

[i.e., take $u = \Gamma$, $v = \Phi$, in Equation (3)], we obtain

$$\int_{\partial D} \left[\Gamma(P, Q') \frac{\partial \Phi}{\partial n_{Q'}} - \Phi(Q') \frac{\partial \Gamma(P,Q')}{\partial n_{Q'}} \right] dS_{Q'} = \int_{D} \Phi(P') \delta_P(P') d\tau_{P'}$$

$$= \Phi(P), \text{ if } P \in \text{int. } \partial D \qquad (4)$$

$$= 0 \quad \text{if} \quad P \in \text{ext. } \partial D \ .$$

The assumptions we have made in applying the Green's theorem are that ∂D is closed and smooth in the sense that ∂D has a continuously varying normal. Under these conditions, we can assign the value of $\frac{1}{2}\Phi(P)$ for the above integral for a point P lying on the boundary ∂D. Since for $P \in \partial D$ the integrand in the surface integral is singular, this value of $\frac{1}{2}\Phi(P)$ for $P \in \partial D$ would mean that the surface integral has a principal value.

Thus Equation (4) provides an integral representation for the solution of (1). Alternatively, we could consider the function defined by

$$\Phi(P) = \int_{\partial D} \left[\Gamma(P, Q') \, \sigma(Q') - \frac{\partial \Gamma(P,Q')}{\partial n_{Q'}} \mu(Q') \right] dS_{Q'} , \qquad (5)$$

where σ and μ are arbitrary functions, continuous on ∂D. By the same argument as before, we see that the above function, Φ, satisfies Equation (1), with the conditions,

$$\Phi(Q') = \mu(Q') \ ,$$

$$\frac{\partial \Phi}{\partial n_{Q'}} = \sigma(Q') \ , \quad Q' \in \partial D \ . \qquad (6)$$

We, therefore, define the single and double layer potential functions, respectively as:

$$V(P) = \int_{\partial D} \sigma(Q') \, \Gamma(P, Q') dS_{Q'} , \tag{7.1}$$

$$W(P) = \int_{\partial D} \mu(Q') \frac{\partial \Gamma(P, Q')}{\partial n_{Q'}} dS_{Q'} . \tag{7.2}$$

It follows from Equations (6) and (7) that the single layer potential V(P) is useful in solving the Laplace equation when the normal derivative of the potential is prescribed on the boundary (Neumann Problem) while the double layer potential W(P) can be employed when the potential is prescribed on the boundary (Dirichlet Problem). It is easily seen from equation (7.1), that

$$\nabla^2 V(P) = 0, \text{ for } P \in D .$$

However, the behavior of the potential on the boundary is not evident. We shall, therefore, investigate V(Q) for $Q \in \partial D$. Let $V_i(Q)$ denote the limiting value of V taken at a sequence of points $\{Q_k\}$ in the region interior to ∂D and tending to $Q \in D$ from within. Let $V_e(Q)$ be the external limit defined similarly. Let us indent the axisymmetric surface, S, in the neighborhood of the singular point, Q, by a semi-toroidal surface of cross sectional radius ε. [Figure 2]. Now the principal value of the surface integral (7.1) is the limit as $\varepsilon \to 0$ of the integral over the indented surface. This integral consists of a contribution due to the integration of $\sigma\Gamma$ over the semi-toroidal surface. If Q' lies on this torus, we have

$$\Gamma(Q,\ Q') \;=\; \frac{1}{4\pi^2 r}\cdot \ln\,\varepsilon\ ,$$

where r is the radius of the body at Q. We then have the contribution from this part to the integral (7.1) as

$$\frac{\sigma(Q)\ln\,\varepsilon}{4\pi^2 r}\int dS \;=\; \frac{\sigma(Q)}{4\pi^2 r}\cdot \ln\,\varepsilon\; 2\pi^2\varepsilon r$$

The above quantity tends to zero as $\varepsilon \to 0$. A similar argument applies when the external limit is taken. We, therefore, have

$$V_i(Q) \;=\; V_e(Q) \;=\; V(Q)\quad . \tag{8}$$

Repeating the limiting process for $\frac{\partial V}{\partial n}$ [by indenting the surface towards the interior of ∂D], we find that the contribution from $\sigma(Q')\frac{\partial \Gamma}{\partial n_{Q'}}$ over the semi-torous is given by

$$\lim_{\varepsilon\to 0}\cdot\frac{\sigma\,(Q)}{4\pi^2\varepsilon r}\int dS \;=\; \lim_{\varepsilon\to 0}\frac{\sigma\,(Q)}{4\pi^2\varepsilon r}\cdot 2\pi^2\varepsilon r \;=\; \frac{\sigma\,(Q)}{2}\quad .$$

Similarly, when the indentation is towards the exterior of ∂D, the above limit is $-\frac{\sigma\,(Q)}{2}$, we therefore obtain

$$\left(\frac{\partial V}{\partial n_Q}\right)_i \;=\; \frac{\sigma\,(Q)}{2} + \frac{\partial V}{\partial n_Q}\ , \tag{9.1}$$

$$\left(\frac{\partial V}{\partial n_Q}\right)_e \;=\; -\,\frac{\sigma\,(Q)}{2} + \frac{\partial V}{\partial n_Q}\quad . \tag{9.2}$$

We conclude, therefore, that the potential due to a source layer (single layer) is continuous across the layer while the normal derivative experiences a jump equal to the source density. We also observe that in the Neumann problem, the interior limit $\left(\frac{\partial V}{\partial n_Q}\right)_i$ of equation (9.1) is specified and this leads to the

integral equation central to our analysis.

2.2 Integral Equation

In this section we solve the Laplace equation for the region between the body and the cylindrical tunnel by means of a surface source distribution. We also examine the boundary conditions at the various surfaces that form the boundary of the region. We note that the boundary ∂D consists of the surfaces S_1, S_2, S_3 and S_4 [Figure 1]. The surfaces S_3 and S_4 are at a large distance fore and aft of the body and we will take the limiting value as this distance tends to infinity.

Let us consider the potential due to a source layer:

$$V(P) = \int_{\partial D} \sigma(Q')\Gamma(P, Q')dS_{Q'} \quad . \tag{7.1}$$

Suppose the boundary condition

$$h(Q) = -\left(\frac{\partial V}{\partial n_Q}\right)_i$$

is specified on ∂D. On S_1, we have

$$h(Q) = h(s) \quad , \tag{10}$$

where $Q \in S_1$ and s is the arc length parameter on S_1. On S_2, we have

$$h(Q) = 0 \quad , \quad Q \in S_2 \quad .$$

Suppose we require

$$\sigma(Q) = 0, \quad \frac{\partial \Gamma}{\partial n_Q}(Q, Q) = 0 \tag{11}$$

for $Q \in S_2$, then the boundary condition on S_2 is satisfied.

On S_3 and S_4, we want the perturbation velocity to become zero as $z \to \pm\infty$. We, therefore, have

$$h(Q) = 0 \quad , \quad Q \in S_3, S_4 \quad . \tag{12}$$

Taking $\sigma(Q) = 0$ for $Q \in S_3$, S_4, we obtain the following constraints on S_3 and S_4. On S_3 we have

$$h(Q) = -\left(\frac{\partial V}{\partial n_Q}\right)$$

$$= -\int_{S_1} \sigma(Q') \frac{\partial \Gamma}{\partial n_Q}(Q, Q') dS_{Q'} \quad .$$

Therefore,

$$h(Q) = \int_{S_1} \sigma(Q') \left\{ \lim_{z \to -\infty} \frac{\partial \Gamma}{\partial z}(Q, Q') \right\} dS_{Q'} \quad , \quad Q \in S_3 \quad . \tag{13}$$

Similarly on S_4, we have

$$h(Q) = -\left(\frac{\partial V}{\partial n_Q}\right)$$

$$= -\int_{S_1} \sigma(Q') \frac{\partial \Gamma(Q, Q')}{\partial n_{Q'}} dS_{Q'} \quad ,$$

or

$$h(Q) = -\int_{S_1} \sigma(Q') \left\{ \operatorname{Lt.}_{z \to \infty} \frac{\partial \Gamma}{\partial z}(Q, Q') \right\} dS_{Q'} \quad , \quad Q \in S_4 \quad . \tag{14}$$

It follows, therefore, from Equations (12), (13) and (14)

$$\int_{S_1} \sigma(Q') \{\lim_{z \to \pm \infty} \frac{\partial \Gamma}{\partial z} (Q; Q')\} dS_{Q'} = 0 \quad . \tag{14.1}$$

The Neumann problem is solved if we obtain the source distribution $\sigma(Q)$ on the body surface. This is obtained from the boundary condition on the body. Let α be the angle between the z-axis and the normal to the body pointing into the fluid [Figure 2]. Then

$$\frac{\partial \Gamma}{\partial n_Q} = -\left(\frac{\partial \Gamma}{\partial z} \cos\alpha + \frac{\partial \Gamma}{\partial r} \sin\alpha\right) ,$$

$$h(Q) = h(s) = -\left(\frac{\partial V}{\partial n_Q}\right)_i .$$

From equation (9.1),

$$-h(s) = \frac{\sigma(s)}{2} - \int_0^{\ell} \sigma(Q') \left(\frac{\partial \Gamma}{\partial z} \cos\alpha + \frac{\partial \Gamma}{\partial r} \sin\alpha\right) . \, 2\pi r' ds' ,$$

or

$$h(s) = -\frac{\sigma(s)}{2} + 2\pi \int_0^{\ell} \sigma(s') \left(\frac{\partial \Gamma}{\partial z} \cos\alpha + \frac{\partial \Gamma}{\partial r} \sin\alpha\right) r' ds' \quad . \tag{15}$$

The above integral equation should be solved for the source strength distribution $\sigma(s)$. Here s is the arc length parameter and ℓ is the total arc length of the body along the meridian. The source strength $\sigma(s)$ must satisfy the supplementary condition (14.1). This supplementary condition is essential for the existence of the solution to the Neumann Problem

and also is of physical significance. We shall return to this point later in the analysis.

2.3 Neumann Function

It remains now to find the Neumann function, $\Gamma(P, P')$ of the above analysis. Since $V(P)$ is the potential due to a source layer (single layer) and the region is axisymmetric, $\Gamma(P, P')$ has the physical interpretation of the potential due to a source ring at P', namely a ring centered at z' and of radius r' kept inside an infinitely long tunnel of radius a.

Let us enumerate the conditions on $\Gamma(P, P')$:

$$\nabla^2\Gamma(P, P') = -\delta_P(P') \quad , \tag{2}$$

$$\frac{\partial\Gamma}{\partial r} = 0 \text{ for } r = a \quad ,$$

$$\frac{\partial\Gamma}{\partial z} = \mp \frac{1}{2\pi a^2} \text{ for } z \to \pm\infty \quad .$$

Since $\Gamma(P, P')$ is a source of unit strength, the mass flow consideration gives us

$$\int_0^a \left(-\frac{\partial\Gamma}{\partial z}(P, P')\right) 2\pi r dr = \frac{1}{2} \qquad z > z' \quad ,$$

$$= \frac{-1}{2} \qquad z < z' \quad .$$

We express $\Gamma(P, P')$ as the sum of two parts: the singular part $N(P, P')$ and the regular part $g(P, P')$, i.e.,

$$\Gamma(P, P') = N(P, P') + g(P, P') \quad ,$$

with

$$\nabla^2 N(P, P') = -\delta_P(P') \quad , \tag{16}$$

$$\frac{\partial N}{\partial r} = 0 \text{ for } r = a \quad , \tag{16.1}$$

$$\int_0^a \frac{\partial N}{\partial z} \cdot 2\pi r dr = 0 \quad , \tag{16.2}$$

$$\frac{\partial N}{\partial z} = 0 \text{ for } z \to \pm\infty \quad , \tag{16.3}$$

and

$$\nabla^2 g(P, P') = 0 \quad , \tag{17}$$

$$\frac{\partial g}{\partial r} = 0 \text{ for } r = a \quad , \tag{17.1}$$

$$\int_0^a \left(-\frac{\partial g}{\partial z}\right) 2\pi r dr = \frac{1}{2} \qquad z > z' , \tag{17.2}$$

$$= \frac{-1}{2} \qquad z < z' ,$$

$$\frac{\partial g}{\partial z} = \mp \frac{1}{2\pi a^2} \text{ for } z \to \pm\infty \quad . \tag{17.3}$$

Let us obtain the solution of Equation (16). Expressing the Laplacian and the Dirac delta operators in cylindrical co-ordinates, we have

$$N_{rr} + \frac{1}{r} N_r + N_{zz} = \frac{-\delta(r - r')}{2\pi r} \, \delta(z - z') \quad . \tag{18}$$

Let us define the finite Hankel transform [17],

$$F_o(k) = \int_0^a f(r) \, J_o(kr) r dr \quad , \tag{19}$$

for the sequence of k, viz. k_1, k_2, ..., given by the roots of the equation

$$J_o'(ka) = 0 \quad . \tag{19.1}$$

The inverse transform for (18) is given by [17],

$$f(r) = \frac{2}{a^2} \sum_k \frac{F_o(k)\, J_o(kr)}{J_o^{\,2}(ka)} \quad . \tag{20}$$

We also have the result [17],

$$\int_0^a (\Delta_o f)\, J_o(kr) r dr = a\, J_o(ka)\, f_r \Big|_{r = a} - k^2 F_o(k) \quad , \tag{21}$$

where Δ_o is the operator given by

$$\Delta_o f = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) f \quad .$$

Let us denote the transform of N by $\tilde{N}$, i.e.

$$\tilde{N}(z;\, r',\, z';\, k) = \int_0^a N(r,z\, ;\, r',z\, ')\, J_o(kr) r dr \quad . \tag{22}$$

Now, applying the finite Hankel transform to equation (18), we obtain

$$aJ_o(ka)\, (N_r)\Big|_{r = a} - k^2\, \tilde{N} + \tilde{N}_{zz} = -\delta(z - z') \int_0^a \frac{\delta(r - r')}{2\pi r}\, J_o(kr) r dr \quad ,$$

$$= -\delta(z - z') \cdot \frac{J_o(kr')}{2\pi} \quad .$$

Invoking the Boundary Condition

$$\frac{\partial N}{\partial r} = 0 \text{ for } r = a, \quad , \tag{16.1}$$

we obtain the ordinary differential equation

$$\tilde{N}_{zz} - k^2 \tilde{N} = -\, \delta(z - z')\, \frac{J_o(kr')}{2\pi} \quad . \tag{23}$$

In order to satisfy the condition at infinity, viz. equation (16.3), we obtain the solution in the form,

$$\tilde{N} = A_1 e^{-k(z - z')} \quad , \qquad z > z'$$

$$\tilde{N} = A_2 e^{k(z - z')} \quad , \qquad z < z'$$

The requirements on $\tilde{N}$ are that $\tilde{N}$ be continuous and the derivative $\frac{d\tilde{N}}{dz}$ have a discontinuity $\frac{-J_o(kr')}{2\pi}$ at $z = z'$. Thus $A_1 = A_2$ and

$$\frac{d\tilde{N}_R}{dz}\Bigg|_{z = z'_+} - \frac{d\tilde{N}}{dz}\Bigg|_{z = z'_-} = -2kA_1 = \frac{-J_o(kr')}{2\pi}$$

or,

$$A_1 = \frac{J_o(kr')}{4\pi} \; .$$

We therefore have

$$\tilde{N}(z;\, z',\, r';\, k) = \frac{J_o(kr')}{4\pi} e^{-k|z - z'|} \tag{24}$$

Let us now invert $\tilde{N}$ and the aid of equation (20) to obtain

$$N(r,z;r',z') = \frac{1}{2\pi a^2} \sum_k \frac{e^{-k|z - z'|} J_o(kr) J_o(kr')}{kJ_o{}^2(ka)} \; . \tag{25}$$

We can now easily verify that N satisfies the Equation (.612) and (16.3). Thus

$$\int_0^a \frac{\partial N}{\partial z} 2\pi r dr = -\frac{1}{a^2}\sum_k e^{-k|z - z'|}\cdot Sgn(z - z')\frac{J_0(kr')}{J_0{}^2(ka)}\int_0^a J_0(kr) rdr \; .$$

The integral in the above equation vanishes since

$$\int_0^a J_0(kr) rdr = \left.\frac{rJ_1(kr)}{k}\right|_0^a = \frac{-aJ_0'(ka)}{k} = 0$$

Thus, the function N given by equation (25) satisfies all conditions (16) through (16.3). Finally the solution of equation (17) is given by

$$g(P, P') = -\frac{1}{2\pi a^2}|z - z'| \quad , \tag{26}$$

as can be easily verified.

We, therefore, obtain the Neumann function

$$\Gamma(r, z; r', z') = \frac{1}{2\pi a^2}\left[-|z - z'| + \sum_k \frac{e^{-k|z - z'|} J_0(kr) J_0(kr')}{kJ_0{}^2(ka)}\right] . \tag{27}$$

The source distribution $\sigma(Q)$ can now be obtained from the integral Equation (15) and the Neumann function (27). We then get

$$h(s) = -\frac{1}{2}\sigma(s) + \int_0^\ell \sigma(s)\left[F_1(s, s')\cos\alpha + F_2(s, s')\sin\alpha\right] r'ds' \quad ,\tag{28}$$

where we have defined

$$F_1(s, s') = 2\pi \frac{\partial \Gamma}{\partial z}(z, r; z', r') ,$$

$$F_2(s, s') = 2\pi \frac{\partial \Gamma}{\partial r}(z, r; z', r') .$$

It follows from Equation (27) that

$$F_1(s, s') = -\frac{1}{a^2} \ \mathrm{Sgn}(z - z') \left[1 + \sum_k \frac{e^{-k|z - z'|} J_o(kr)\, J_o(kr')}{J_o^{\,2}(ka)} \right] , \quad (29.1)$$

$$F_2(s, s') = -\frac{1}{a^2} \sum_k \frac{e^{-k|z - z'|} J_1(kr)\, J_o(kr')}{J_o^{\,2}(ka)} . \quad (29.2)$$

It should be noted that the source distribution given by equations (28) and (29) should satisfy the condition (14.1). This is an essential condition for the existence of the solution to the Neumann problem. It is known from potential theory that the solution to the Neumann problem on a bounded domain D exists only when the integral, taken over the bounding surface, of the values assigned to the normal derivative vanishes [18]. As applied to fluid mechanics, this is the continuity condition (or conservation of mass) for the control volume D. Rewriting the condition (14.1) with the use of equation (27) we get

$$\int_0^{\ell} \sigma(s') r' ds' = 0 , \quad (30)$$

where s' is the arc length parameter on the body and ℓ is the total arc length of the body along the meridian. The condition (30) states that the areal sum of sources over the body should be zero. We note that this condition is automatically satisfied by a closed body. We shall present here a simple proof based on physical reasoning: it follows from our analysis that half the volume of fluid from the source layer on the body flows out

of the body while the other half flows into the body. Since there is no flow across the body surface and the body is closed, the net flow into the body must be zero. Otherwise, the continuity condition for the region inside the body would be violated. Therefore, for a closed body, the areal sum of sources is zero.

2.4 Summation of Series

The infinite series in Equations (29.1) and (29.2) are rapidly convergent when the axial distance from the source, $|z - z'|$, is large. However, convergence is slow near the singularity and the series in Equation (29.1) does not converge at all for $z = z'$. These defects can be remedied by expressing these series in terms of elliptic integrals and convergent series involving $|z - z'|$ and elementary functions of r and r'. This part of the investigation presents quite a few features of mathematical interest and is an exercise in itself. Therefore, only the results of the investigation are presented here and the reader can consult Appendix A for details.

Let us define the following sums:

$$\Xi_1 = \sum_k \frac{e^{-kx} J_0(kr)\, J_0(kr')}{J_0{}^2(ka)} , \tag{31}$$

$$\Xi_2 = \sum_k \frac{e^{-kx} J_1(kr)\, J_0(kr')}{J_0{}^2(ka)} . \tag{32}$$

Then for small values of $x > 0$, the following approximations are valid:

$$\begin{aligned} \Xi_1 = -1 &+ \frac{a^2 x E(m)}{\pi(1 - m^2)\,[x^2 + (r + r')^2]^{3/2}} + \frac{\mu_2 x}{a} \\ &+ \frac{\mu_4 x}{a^3}\left(\frac{P_2}{4} - \frac{x^2}{6}\right) \\ &+ \frac{\mu_6 x}{a^5}\left(\frac{P_4}{64} - \frac{P_2 x^2}{24} + \frac{x^4}{120}\right) \end{aligned}$$

$$+ \frac{\mu_8 x}{a^7}\left(\frac{P_6}{2304} - \frac{P_4 x^2}{384} + \frac{P_2 x^4}{480} - \frac{x^2}{5040}\right)$$

$$+ 0\left(\frac{1}{a^9}\right) . \tag{33}$$

$$\Xi_2 = \frac{a^2}{2\pi r[x^2 + (r + r')^2]^{\frac{1}{2}}} \left[\frac{(r^2 - r'^2 - x^2)\, E(m)}{x^2 + (r - r')^2} + K(m)\right]$$

$$- \frac{\mu_2 Q_o}{2a} - \frac{\mu_4}{a^3}\left(\frac{Q_2}{16} - \frac{Q_o x^2}{4}\right)$$

$$- \frac{\mu_6}{a^5}\left(\frac{Q_4}{384} - \frac{Q_2 x^2}{32} + \frac{Q_o x^4}{48}\right)$$

$$- \frac{\mu_8}{a^7}\left(\frac{Q_6}{18432} - \frac{Q_4 x^2}{768} + \frac{Q_2 x^4}{384} - \frac{Q_o x^6}{1440}\right)$$

$$- 0\left(\frac{1}{a^9}\right) . \tag{34}$$

The functions K(m) and E(m) of the above equations are the complete elliptic integrals of the first and second kind respectively. The modulus, m, of these elliptic integrals is given by

$$m^2 = \frac{4rr'}{x^2 + (r + r')^2} . \tag{35}$$

The P_n and Q_n of equations (33) and (34) are functions of r and r'. These functions and the coefficients μ_n are given in Appendix A.

The equations (33) and (34) contain an important feature of our solution. They enable us to obtain the flow field in an unbounded medium (no tunnel wall) as the limit $a \to \infty$ of the flow inside the tunnel. Using equations (29.1), (29.2), (33) and (34) we obtain the following limits:

$$\lim_{a \to \infty} F_1(s, s') = - \frac{E(m)\ (z - z')}{\pi(1 - m^2)\ [(z - z')^2 + (r + r')^2]^{3/2}} , \tag{36}$$

$$\lim_{a \to \infty} F_2(s, s') = -\frac{1}{2\pi r[(z - z')^2 + (r + r')^2]^{1/2}} \times \left[\frac{(r^2 - r'^2 - x^2)\, E(m)}{(z - z')^2 + (r - r')^2} + K(m)\right]. \quad (37)$$

The modulus, m, is now given by Equation (35) with x replaced by (z - z′). The reader can verify that the solution for an unbounded medium is indeed given by the integral equation (28) along with Eqs. (36) and (37), [19].

There are various techniques for the solution of the Fredholm integral equation of the second kind; namely, Eq. (28), [18]. One method of solution is to discretize the integral equation using some form of numerical quadrature scheme, Simpson's rule for example. This technique, also known as algebraization, reduces Equation (28) to a matrix equation in the unknowns $\{\sigma(s_i), i = 1, \ldots N\}$ where N is the number of intervals chosen. The resulting matrix equation will show strong diagonal dominance which ensures computational stability. (This is the same reason why an integral equation of the second kind is preferable to that of the first kind.)

2.5 Numerical Example

The integral equation (28) has been solved numerically by the algebraization method. In this method, the body surface is divided into N segments. The segments are assumed to be of parabolic shape and the source strength is assumed to vary linearly on each element. Simpson's rule is used for the integration indicated in equation (28). This procedure discretizes Eq. (28) into a matrix equation in the unknowns, $\{\sigma(s_i), i = 1, 2, \ldots N\}$, where $\sigma(s_i)$ is the source strength at the mid-point of the i-th segment. Once the singularity distribution is obtained, the velocities and the pressure distribution on the body (or at any point in the flow field) are easily obtained. The computational method has been programmed in Fortran IV language for the IBM 370/3033

processor at the Pennsylvania State University.

The exact solution for the Rankine ovoid was employed to check the calculations. For a given tunnel radius, maximum body radius and the source locations, the shape of the ovoid was obtained by using Lamb's results [1]. The poor convergence of the infinite series given by Lamb was overcome by using Watson's results [3] in the neighbourhood of the singularities. The resulting shapes for a family of ovoids is shown in Figure 3. The present surface singularity method was employed to analyze the ovoids, which were divided into 48 elements. The pressure distribution obtained by the present method shows excellent agreement with the analytical solution [Figure 3] in all cases. The present method has also been used for various body shapes of interest. These results along with the details of the numerical solution will appear in a companion memorandum.

APPENDIX A

Summation of Series of Bessel Functions

I. Define

$$\Xi_1 = \sum_{m=1}^{\infty} \frac{J_0(j_m r/a)\, J_0(j_m r'/a) e^{-j_m \cdot x/a}}{J_2^2(j_m)} \tag{A-1}$$

where j_1, j_2, ... are the positive zeros of $J_1(w)$.

Consider the function,

$$\frac{[J_1(w)Y_0(rw/a) - J_0(rw/a)Y_1(w)]}{J_1(w)} \cdot \frac{\pi}{2} \cdot wJ_0(r'w/a)e^{-xw/a} . \tag{A-2}$$

The above function has simple poles at j_m with residue

$$-\frac{J_0(rj_m/a)Y_1(j_m) \cdot \frac{\pi}{2} \cdot j_m \cdot J_0(r'j_m/a)e^{-\frac{x}{a}j_m}}{J_1'(j_m)} \tag{A-3}$$

From equation (12), Section (3.63), of Reference [20],

$$J_1(z)Y_2(z) - J_2(z)Y_1(z) = \frac{-2}{\pi z} .$$

Since $J_1(j_m) = 0$ we have $Y_1(j_m)j_m = \frac{2}{\pi J_2(j_m)}$ (A-4)

Also, $J_1'(j_m) = -J_2(j_m)$ (A-5)

Substituting (A-4) and (A-5) into (A-3), it follows that the terms of series are the residues at j_m of function (A-2). By Cauchy's theorem, the sum of residues is

$$\Xi_1 = -\frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{[J_1(w)Y_0(rw/a) - J_0(rw/a)Y_1(w)]}{J_1(w)} \frac{\pi}{2} \cdot we^{-xw/a} J_0(r'w/a)dw.$$

where the contour of integration consists of a semicircle of infinite radius lying on the right half of the w-plane and the imaginary axis indented to the

right of the origin by a small radius thereby excluding the simple pole at the origin. It is easy to verify that the residue of the integrand at the origin (*) is 2. Therefore, the contribution to Ξ_1 from the semi-circular indentation around the origin is -1. Also, the contribution from the infinite semicircle vanishes. Therefore the contribution from the imaginary axis alone remains to be evaluated.

Substituting $w = \pm it$ for the two halfs of the imaginary axis, and adding the corresponding integrals we obtain

$$\Xi_1 = -1 + \frac{1}{\pi} \int_0^\infty \frac{I_1(t)K_0(rt/a) + I_0(rt/a)K_1(t)}{I_1(t)} \, t \sin(\frac{xt}{a}) I_0(\frac{r't}{a}) \, dt \, . \quad \text{(A-6)}$$

First let us consider the integral

$$G_1 = \frac{1}{\pi} \int_0^\infty K_0(rt/a) I_0(r't/a) \sin(xt/a) t \, dt \quad . \quad \text{(A-7)}$$

The above integral can be expressed in terms of elliptic integrals by using Kirchoff's formula [21], viz.

$$\int_0^\infty K_0(bt) I_0(ct) \cos qt \, dt = \int_0^{\pi/2} \frac{d\theta}{[\, q^2 + (b+c)^2 - 4bc \sin^2 \theta \,]^{1/2}} \, . \quad \text{(A-8)}$$

Differentiating the above equation with respect to the parameter q (and interchanging the order of integration and differentation) we obtain

$$\int_0^\infty K_0(bt) I_0(ct) \sin qt \; t \, dt = q \int_0^{\pi/2} \frac{d\theta}{[q^2 + (b+c)^2 - 4bc \sin^2 \theta]^{3/2}} \, ,$$

* The following results [Reference 25] are found useful in studying the behavior of the various integrands at the origin. When ν is fixed, as $z \to 0$, we have

$$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1) \qquad Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(\nu) (\frac{1}{2}z)^{-\nu}$$
$$Y_0(z) \sim \frac{2}{\pi} \ln z$$

$$= \frac{q}{[q^2 + (b+c)^2]^{3/2}} \int_0^{\pi/2} \frac{d\theta}{[1 - m^2 \sin^2\theta]^{3/2}} , \quad \text{(A-9)}$$

where $m^2 = \dfrac{4bc}{q^2+(b+c)^2}$.

Taking $b = r/a$, $c = r'/a$, $q = x/a$, the integral G_1 becomes,

$$G_1 = \frac{a^2 x \quad E(m)}{\pi[x^2 + (r+r')^2]^{3/2} \, (1-m^2)} , \quad \text{(A-10)}$$

with

$$m^2 = \frac{4rr'}{x^2 + (r+r')^2} .$$

Next let us evaluate the integral

$$G_2 = \frac{1}{\pi} \int_0^\infty \frac{K_1(t)}{I_1(t)} \; I_0(rt/a) \; I_0(r't/a) \; \sin(xt/a)t \; dt \quad . \quad \text{(A-11)}$$

The product of the two I_0-functions can be replaced by the Sonine-Gegenbaur Integral [22], viz.

$$I_0(z)I_0(z') = \frac{1}{\pi} \int_0^\pi I_0(\sqrt{z^2 + z'^2 - 2zz' \cos\gamma}) \; d\gamma \quad . \quad \text{(A-12)}$$

Taking $z = r/a$, $z' = r'/a$ and $R^2 = r^2 + r'^2 - 2rr'\cos\gamma$ and interchanging the order of integration, we get

$$G_2 = \frac{1}{\pi^2} \int_0^\pi d\gamma \int_0^\infty \frac{K_1(t)}{I_1(t)} \; I_0(\frac{Rt}{a}) \; \sin(\frac{xt}{a}) \; t \; dt \quad . \quad \text{(A-13)}$$

The infinite integral with respect to t in the above equation has been evaluated by Watson [3] for values of R and x satisfying the condition

$$R^2 + x^2 < 4a^2. \qquad \text{(A-14)}$$

Watson's investigation is elaborate and consists of expanding the product

$$I_0(Rt/a)\ \sin(xt/a)$$

in ascending powers of t/a and integrating the resulting series term by term. The reader should consult Reference [3] for further details in this regard. Using Watson's result and carrying out the integration with respect to γ, we obtain

$$\Xi_1 = -1 + \frac{a^2 x\, E(m)}{\pi(1-m^2)[x^2 + (r+r_1)^2]^{3/2}} + \frac{\mu_2 x}{a} + \frac{\mu_4 x}{a^3}\left(\frac{P_2}{4} - \frac{x^2}{6}\right)$$

$$+ \frac{\mu_6 x}{a^5}\left(\frac{P_4}{64} - \frac{P_2 x^2}{24} + \frac{x^4}{120}\right) + \frac{\mu_8 x}{a^7}\left(\frac{P_6}{2304} - \frac{P_4 x^2}{384} + \frac{P_2 x^4}{480} - \frac{x^6}{5040}\right)$$

$$+ O(1/a^9)\ . \qquad \text{(A-15)}$$

The coefficients μ_{2n} in the above equation are defined as

$$\mu_{2n} = \frac{1}{\pi}\int_0^\infty t^{2n}\,\frac{K_1(t)}{I_1(t)}\,dt\ , \qquad \text{(A-16)}$$

and are evaluated in Reference [3]. The P_n's are functions of the radii r and r_1 and are defined as

$$P_n(r,r') = \frac{1}{\pi}\int_0^\pi (r^2 + r'^2 - 2rr'\cos\gamma)^{n/2}\,d\gamma. \qquad \text{(A-17)}$$

The functions P_n will be evaluated in section III.

The integrals μ_{2n} of equation (A-16) which are of practical importance, were first studied by Watson in Reference [3]. Watson Expressed these integrals in terms of a rapidly convergent infinite series. Subsequently Smythe [23] encountered these integrals in his analysis of the flow around a sphere in a circular tube and evaluated them by a numerical quadrature scheme using Weddle's rule. Smythe's values for these integrals were also employed by Miloh [15] in his study of the flow past a prolate spheroid in a tube. The following values were computed by this author for the μ_{2n}'s by Watson's method.

$$\mu_2 = 0.7968241731 \qquad \mu_6 = 7.4582940625$$
$$\mu_4 = 1.2004703485 \qquad \mu_8 = 96.2205196159 \tag{A-18}$$

II. Consider the sum of the series,

$$\Xi_2 = \sum_{m=1}^{\infty} \frac{J_1(j_m r/a)\, J_0(j_m r'/a) e^{-j_m x/a}}{J_2^{\,2}(j_m)} . \tag{A-19}$$

The terms of the above series are the residues at j_m of the function

$$\frac{[J_1(w)Y_1(rw/a) - J_1(rw/a)Y_1(w)]}{J_1(w)} \cdot \frac{\pi}{2}\, w\, J_0(rw/a)\, e^{-xw/a} . \tag{A-20}$$

It is observed that the residue of the above function at j_m is

$$\frac{-J_1(rj_m/a)\, Y_1(j_m)\, \frac{\pi}{2} \cdot j_m\, J_0(r'j_m/a) e^{-xj_m/a}}{J_1'(j_m)} .$$

Now $J_1'(j_m) = -J_2(j_m)$ and $Y_1(j_m)\, j_m = \frac{2}{\pi J_2(j_m)}$. Substituting this last result into

the above expression proves the assertion (A-20). Therefore by Cauchy's residue theorem

$$\Xi_2 = -\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{J_1(w)Y_1(rw/a) - J_1(rw/a)\,Y_1(w)}{J_1(w)} \cdot \frac{\pi}{2} \cdot wJ_0(r'w/a)e^{-xw/a}\,dw, \quad \text{(A-21)}$$

where the path of integration is the imaginary axis since the integrand is analytic (well behaved) at the origin. Taking $w = \pm it$ on the imaginary axis and combining the integrals on the two parts, we obtain

$$\Xi_2 = \frac{1}{\pi}\int_0^{\infty} \frac{[I_1(t)K_1(rt/a) - I_1(rt/a)K_1(t)]\; I_0(r't/a)\cos(\frac{xt}{a}).\; t\;dt.}{I_1(t)} \quad \text{(A-22)}$$

Consider the integral G_3 given by

$$G_3 = \frac{1}{\pi}\int_0^{\infty} K_1(\frac{rt}{a})\; I_0(\frac{r't}{a})\;\cos(\frac{xt}{a})\;t\;dt\;. \quad \text{(A-23)}$$

The integral G_3 can be evaluated by differentiating Kirchoff's formula, namely equation (A-8) with respect to b. Thus,

$$\int_0^{\infty} K_1(bt)I_0(ct)\cos qt\;t\;dt = \int_0^{\pi/2} \frac{(b+c) - 2\,c\,\sin^2\theta}{[q^2 + (b+c)^2 - 4bc\,\sin^2\theta]^{3/2}}$$

$$= \frac{1}{[q^2 + (b+c)^2]^{3/2}}\left[(b+c)\int_0^{\pi/2}\frac{d\theta}{(1-m^2\sin^2\theta)^{3/2}} - 2c\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(1-m^2\sin^2\theta)^{3/2}}\right]$$

$$= \frac{m^3}{8(bc)^{3/2}}\left[(b+c)\frac{E(m)}{(1-m^2)} - \frac{2c}{m^2}\left\{\frac{E(m)}{(1-m^2)} - K(m)\right\}\right].$$

Taking $b = r/a$, $c = r'/a$ and $q = x/a$ we obtain

$$G_3 = \frac{a^2}{2\pi r[x^2+(r+r')^2]^{\frac{1}{2}}} \left[\frac{r^2 - r'^2 - x^2}{x^2 + (r-r')^2} E(m) + K(m) \right] , \qquad \text{(A-24)}$$

where m is given by equation (A-10).

Next we evaluate the integral

$$G_4 = - \frac{1}{\pi} \int_0^{\infty} \frac{K_1(t)}{I_1(t)} I_1(\frac{rt}{a}) I_0(\frac{r't}{a}) \cos (\frac{xt}{a}) t \, dt \quad . \qquad \text{(A-25)}$$

By the Sonine-Gegenbaur relation,

$$I_0(\frac{rt}{a}) I_0(\frac{r't}{a}) = \frac{1}{\pi} \int_0^{\pi} I_0(Rt/a) \, d\gamma \quad , \qquad \text{(A-12)}$$

where

$$R^2 = r^2 + r'^2 - 2rr' \cos \gamma \, .$$

Differentiating eqn. (A-12) with respect to r,

$$I_1(rt/a) I_0(r't/a) = \frac{1}{\pi} \int_0^{\pi} \frac{I_1(Rt/a)}{R} (r-r' \cos \gamma) \, d\gamma \quad , \qquad \text{(A-26)}$$

and substituting in equation (A-25), then interchanging the order of integration, we get

$$G_4 = - \frac{1}{\pi^2} \int_0^{\pi} \frac{(r-r' \cos \gamma)}{R} d\gamma \int_0^{\infty} \frac{K_1(t)}{I_1(t)} \cdot I_1(Rt/a) \cos(\frac{xt}{a}) t \, dt.$$

We now expand the product $I_1(rt/a) \cos(\frac{xt}{a})\, t$ in ascending powers of t and carry out the two successive integrations. Combining this result with the results (A-22) and (A-24), we obtain

$$\Xi_2 = \frac{a^2}{2\pi r[x^2 + (r + r')^2]^{1/2}} \left[\frac{(r^2 - r'^2 - x^2)E(m)}{x^2 + (r-r')^2} + K(m) \right]$$

$$- \frac{\mu_2 Q_0}{2a} - \frac{\mu_4}{a^3} \left(\frac{Q_2}{16} - \frac{Q_0 x^2}{4} \right) - \frac{\mu_6}{a^5} \left(\frac{Q_4}{384} - \frac{Q_2 x^2}{32} + \frac{Q_0 x^4}{48} \right)$$

$$- \frac{\mu_8}{a^7} \left(\frac{Q_6}{18432} - \frac{Q_4 x^2}{768} + \frac{Q_2 x^4}{384} - \frac{Q_0 x^6}{1440} \right) + O(1/a^9) \qquad \text{(A-27)}$$

The coefficients μ_{2n} have been defined previously. The functions Q_n are defined as

$$Q_n(r, r') = \frac{1}{\pi} \int_0^{\pi} (r - r'\cos\gamma) \, (r^2 + r'^2 - 2rr'\cos\gamma)^{n/2} \, d\gamma . \qquad \text{(A-28)}$$

The functions Q_n will be evaluated in section III.

III. In this section the function $P_n(r,r')$ will be evaluated. By our definition:

$$P_n(r,r') = \frac{1}{\pi} \int_0^{\pi} (r^2 + r'^2 - 2rr' \cos \gamma)^{n/2} \, d\gamma \quad . \text{(A-17)}$$

We can transform the above integral to elliptic or pseudo-elliptic integrals. It is easily seen that

$$r^2 + r'^2 - 2rr' \cos \gamma = r^2 + r'^2 - 2rr'(2 \cos^2 \frac{\gamma}{2} - 1) = (r + r')^2 (1 - h^2 \cos^2 \frac{\gamma}{2}) ,$$

where $$h^2 = \frac{4rr'}{(r+r')^2} .$$

Hence equation (A-17) transforms to

$$P_n(r,r') = \frac{2}{\pi} (r+r')^n H_n , \qquad \text{(A-29)}$$

where H_n is given by the integral

$$H_n = \tfrac{1}{2} \int_0^{\pi} (1 - h^2 \cos^2 \frac{\gamma}{2})^{n/2} \, d\gamma$$

By change of variable, $\gamma = \pi - 2\theta$, we obtain

$$H_n = \int_0^{\pi/2} (1 - h^2 \sin^2 \theta)^{n/2} \, d\theta \qquad \text{(A-30)}$$

The integrals H_n obey the following recurrence relation [24]:

$$H_n = \frac{n-1}{n} (2 - h^2) H_{n-2} - \frac{n-2}{n} (1 - h^2) H_{n-4} \quad . \qquad \text{(A-31)}$$

This recurrence relation requires the values of H_n for two successive even or odd indices depending upon whether n is even or odd. However, our calculations employ only the even values of n. Therefore the following starting values would serve our purpose.

$$H_0 = \pi/2 \qquad \text{(A-32)}$$

$$H_2 = \pi(1-h^2/2)/2$$

It remains to evaluate the functions Q_n. We have

$$Q_n (r,r') = \frac{1}{\pi} \int_0^{\pi} (r-r' \cos \gamma) (r^2 + r'^2 - 2rr' \cos \gamma)^{n/2} \, d\gamma$$

$$= \frac{(r+r')^n}{\pi} \int_0^{\pi} (r + r' - 2r' \cos^2 \gamma/2)(1 - h^2 \cos^2 \gamma/2)^{n/2} \, d\gamma \quad ,$$

$$= \frac{(r+r')^{n+1}}{\pi} \int_0^{\pi} (1-h^2 \cos^2 \frac{\gamma}{2})^{n/2} \, d\gamma - \frac{(r+r')^n}{\pi} . 2r' \int_0^{\pi} \cos^2 \frac{\gamma}{2} (1-h^2 \cos^2 \frac{\gamma}{2})^{n/2} \, d\gamma .$$

Substituting $\gamma = \pi - 2\theta$ we obtain, after some algebraic manipulations,

$$Q_n(r,r') = \frac{(r+r')^{n+1}}{\pi r} \left[(r-r') H_n + (r+r') H_{n+2}\right], \quad \text{(A-33)}$$

where H_n is given by equations (A-31) and (A-32).

It should be noted that the series Ξ_1 and Ξ_2 arose in connection with our work on ring sources while the series of Watson [3] pertain to point sources. It is therefore possible to recover Watson's results when we let the radius of the source ring, namely r', become zero. We then obtain

$$\lim_{r' \to 0} \Xi_1 = S_3$$

and

$$\lim_{r' \to 0} \Xi_2 = a \frac{\partial S_4}{\partial x} \quad \text{(A-34)}$$

where S_3 and S_4 are the series of Reference [3]. The relations (A-34) are easy to verify and therefore their proof is not presented here.

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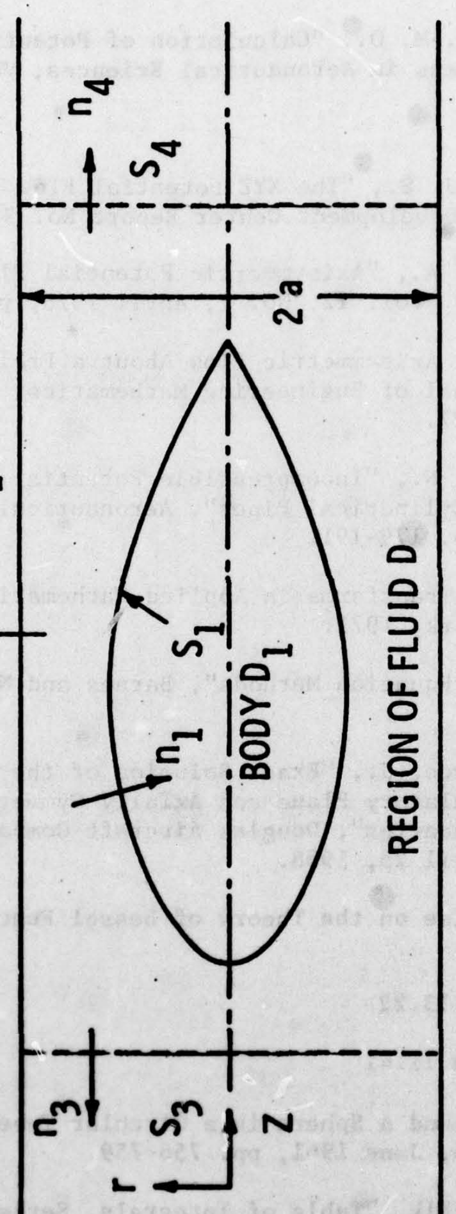

Figure 1. Schematic of Body in a Tunnel

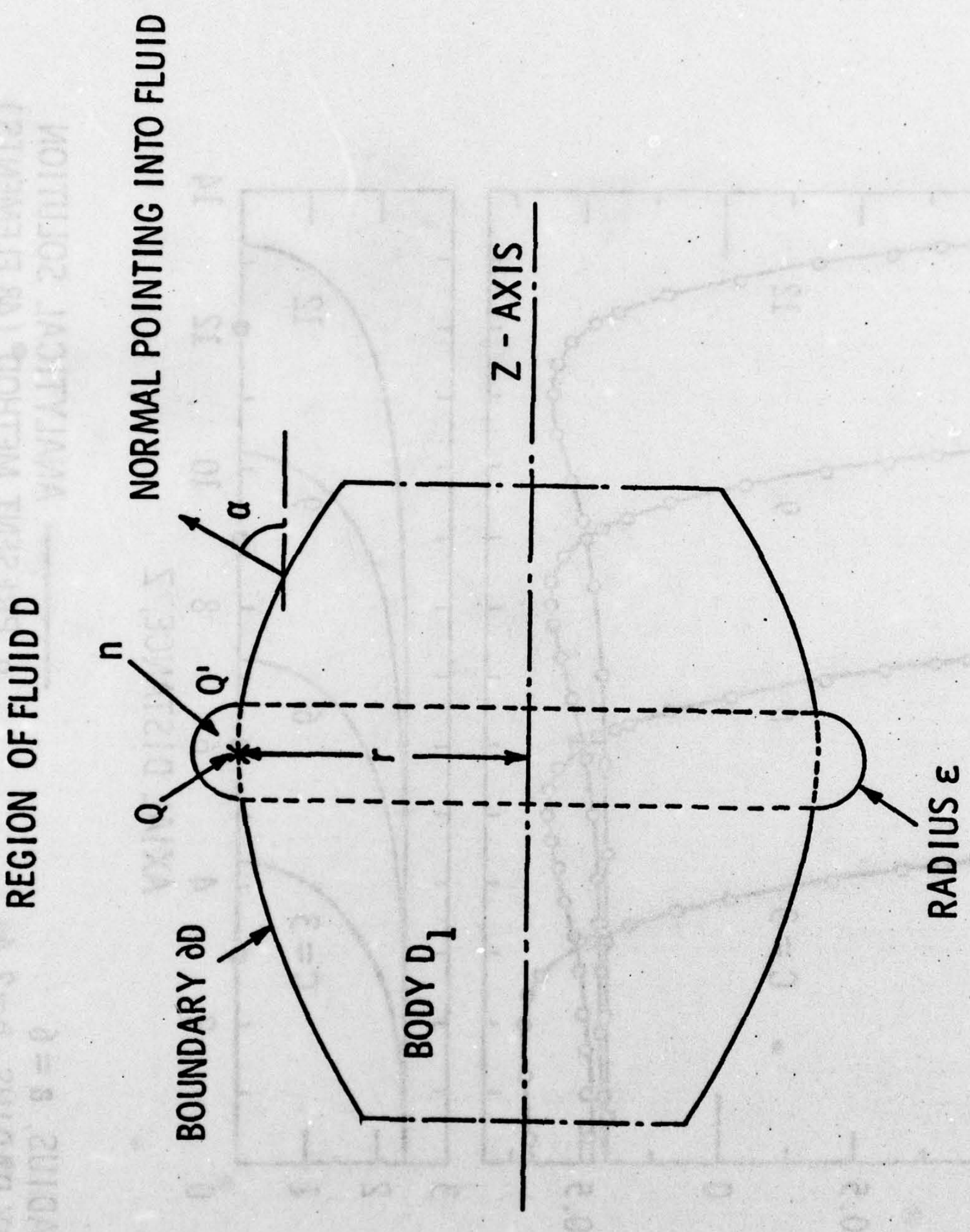

Figure 2. Indentation of the Body

2

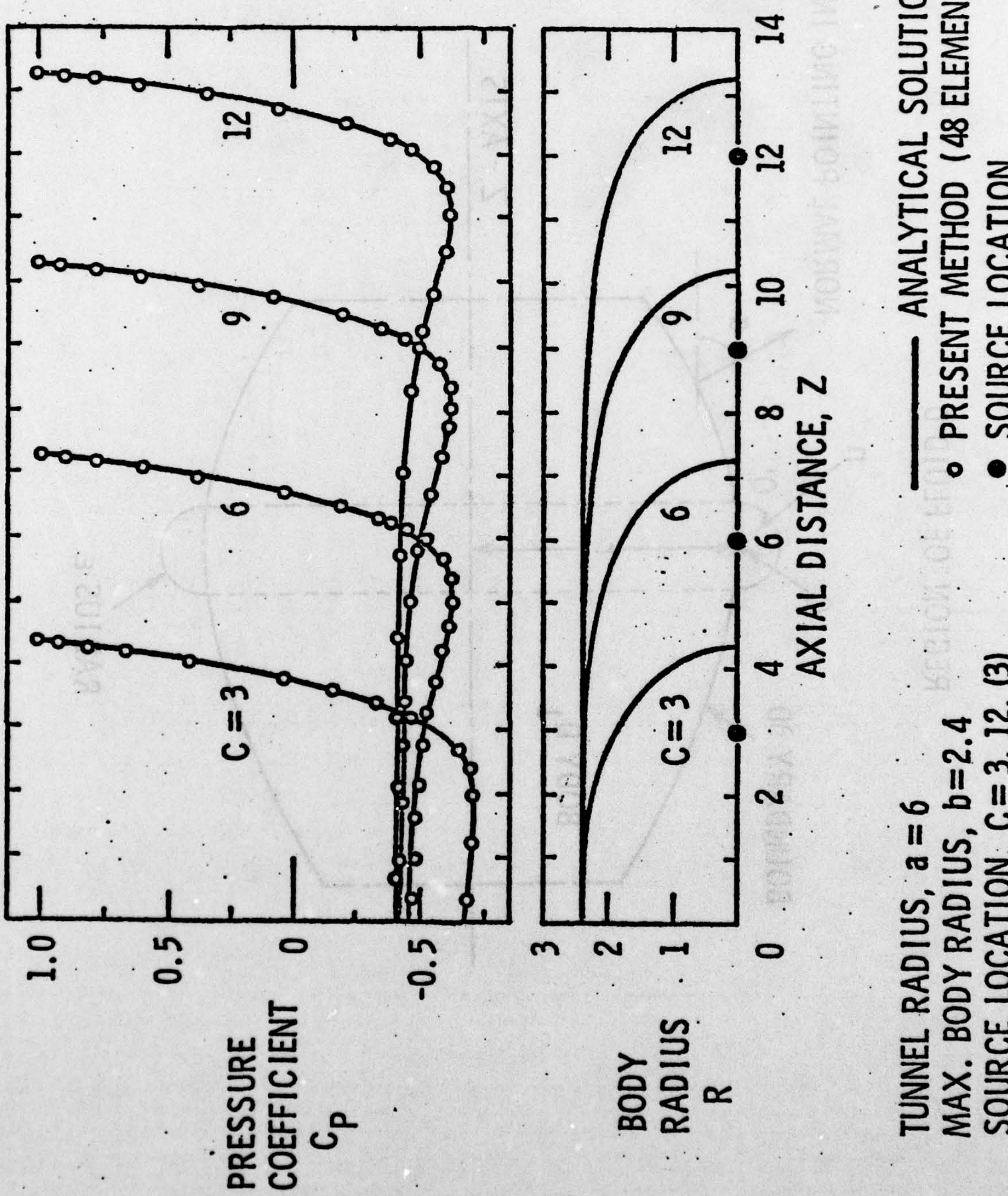

Figure 3. Rankine Ovoids in Tunnel

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